Ellipse Worksheet

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1 Introduction

The equation of an ellipse can be described like this

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

Bringing the second term to the right and squaring both sides, the equation becomes

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

Solving for the square root terms, the equation becomes

$$\sqrt{(x-c)^2+y^2}=-\frac{1}{4a}(x^2+2xc+c^2+y^2-4a^2-x^2+2xc-c^2-y^2)$$
$$\sqrt{(x-c)^2+y^2}=-\frac{1}{4a}(4xc-4a^2)$$
$$\sqrt{(x-c)^2+y^2}=a-\frac{c}{a}x$$

Squaring both sides of the equation, we get

$$x^2-2xc+c^2+y^2=a^2-2cx+\frac{c^2}{a^2}x^2$$

Grouping the x terms, we get

$$x^2\frac{a^2-c^2}{a^2}+y^2=a^2-c^2$$

which cam be written as

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

Let $b^2=a^2-c^2$
The equation can be written in the familiar form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The standard form of an ellipse centered at (h, k) is the equation

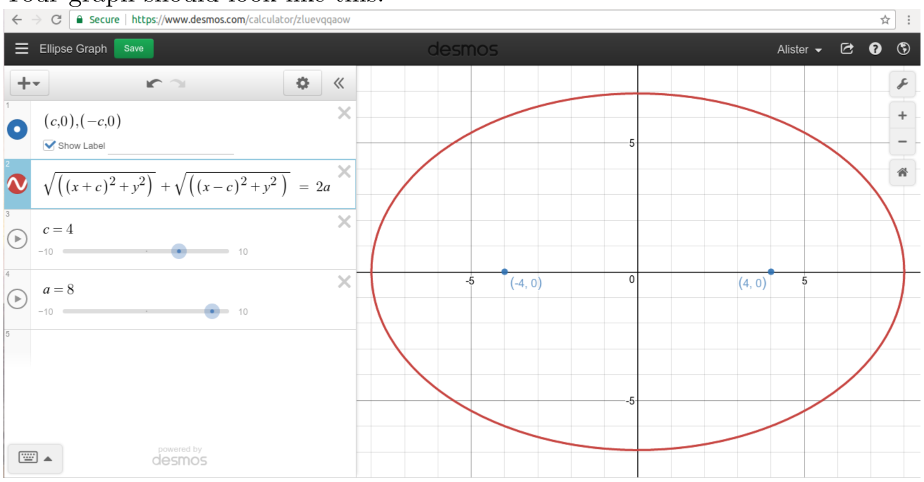
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

a is equal to half the length of the horizontal axis, while b is equal to half the length of the vertical axis, one-half its shortest diameter.

Open Desmos in your computers.
Using Desmos, input the equation

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

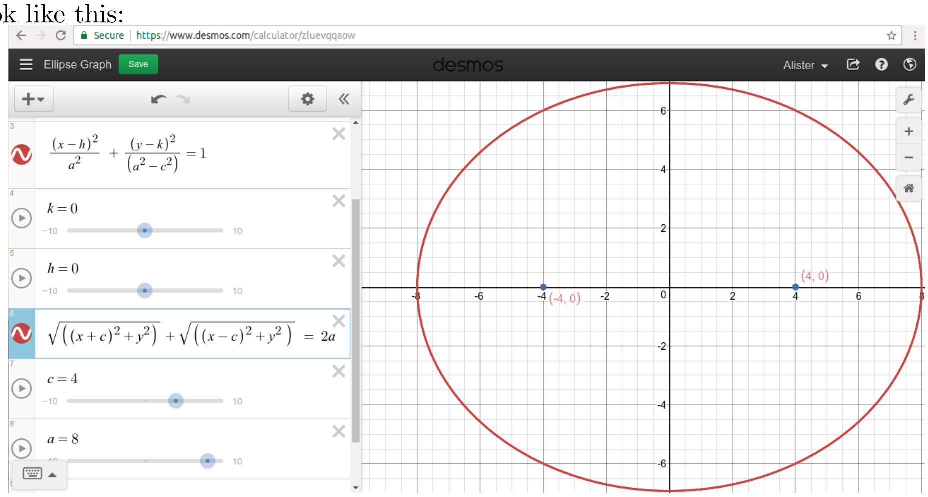
into your Desmos graph. Set a equal to 8 and c equal to 4.
Your graph should look like this.



Next, input the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

, setting h and k equal to 0, into your Desmos graph. Your screen should now look like this:



The equations are exactly the same; adjusting the h and k sliders, the top ellipse moves, while the bottom one stays constant. This is a graphical proof that the equation

$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$$

and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = (a^2 - c^2)$ are equal.

Problems

Problem 1: An ellipse has a center at (0,0), foci at (-4, 0) and (4,0) and the sum of the distances between any point on the ellipse and its foci is 16. What is the equation of the ellipse?

Problem 2: An ellipse has a center at (0,0), foci at (-6, 0) and (6,0) and the sum of the distances between any point on the ellipse and its foci is 18. What is the equation of the ellipse?

Problem 3: An ellipse has a center at (0,0), foci at (-12, 0) and (12,0) and the sum of the distances between any point on the ellipse and its foci is 32. What is the equation of the ellipse?

Problem 4: An ellipse has a center at (1,2). Its horizontal diameter, or major axis is 8 units, and its vertica diameter is 6 units. The major axis is horizontal: the ellipse is wider than it is tall. What is the equation of the ellipse?

Problem 5: An ellipse has a center at (-1,3). Its longest diameter, or major axis is 10 units, and its shortest diameter is 4 units. The major axis is vertical: the ellipse is taller than it is wide. What is the equation of the ellipse?

Problem 6: An ellipse has a center at (2,-3). Its longest diameter, or major axis is 10 units, and its shortest diameter is also 10 units. What is the equation of the ellipse?

Solutions

Solution 1:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

Because $2a$, the sum of the distances between any point on the ellipse and its foci is 16, $a = 8$. The foci of this ellipse are located at $(-c, 0), (c, 0)$; c=4. The equation of our ellipse is

$$\frac{x^2}{8^2} + \frac{y^2}{8^2 - 4^2} = 1$$

or

$$\frac{x^2}{64} + \frac{y^2}{48} = 1$$

Solution 2 Because $2a$, the sum of the distances between any point on the ellipse and its foci is 18, $a = 9$. The foci of this ellipse are located at $(-c, 0), (c, 0)$; c=6. The equation of our ellipse is

$$\frac{x^2}{9^2} + \frac{y^2}{9^2 - 6^2} = 1$$

or

$$\frac{x^2}{81} + \frac{y^2}{45} = 1$$

Solution 3 Because $2a$, the sum of the distances between any point on the ellipse and its foci is 32, $a = 16$. The foci of this ellipse are located at $(-c, 0), (c, 0)$; c=12. The equation of our ellipse is

$$\frac{x^2}{16^2} + \frac{y^2}{16^2 - 12^2} = 1$$

or

$$\frac{x^2}{256} + \frac{y^2}{112} = 1$$

Solution 4 The center of this ellipse is (1,2), so $h = 1$ and $k = 2$. a is equal to half of the length of the horizontal axis, or $(0.5)(8) = 4$. $a = 4$. b is equal to half the length of the vertical axis; $b = 3$. Using the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

with these values, we get

$$\frac{(x-1)^2}{16} + \frac{(y-2)^2}{9} = 1$$

Solution 5 The center of this ellipse is (-1,3), so $h = -1$ and $k = 3$. a is equal to half of the length of the horizontal axis, or $(0.5)(4) = 2$. $a = 2$. b is equal to half the length of the vertical axis; $b = 5$. Using the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

with these values, we get

$$\frac{(x+1)^2}{4} + \frac{(y-3)^2}{25} = 1$$

Solution 6 The center of this ellipse is (-2,3), so $h = -2$ and $k = 3$. a is equal to half of the length of the horizontal axis, or $(0.5)(10) = 5$. $a = 5$. b is equal to half the length of the vertical axis; $b = 5$. Using the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

with these values, we get

$$\frac{(x+2)^2}{25} + \frac{(y-3)^2}{25} = 1$$